The solution of continuous-time algebraic Riccati equations by means of the SOR-LIKE method

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Abstract

The main subject of the paper is to demonstrate the efficiency of the SOR-LIKE method, introduced recently by the author for solving Sylvester equations [1,2], with solving the continuous-time algebraic Riccati equation (CARE)

$$\mathbf{A}^T\mathbf{X} + \mathbf{X}\mathbf{A} - \mathbf{X}\mathbf{S}\mathbf{X} + \mathbf{Q} = \mathbf{0}, \qquad (1)$$

where $\mathbf{A}, \mathbf{Q}, \mathbf{S}, \mathbf{X} \in \mathbb{R}^{n\times n}$ and the matrix $\mathbf{S}$ is usually factored in the following form $\mathbf{S} = \mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T$ with $\mathbf{B} \in \mathbb{R}^{n\times m}$ and $\mathbf{R} \in \mathbb{R}^{m\times m}$. Evidently, when $\mathbf{S}$ is the null matrix, the above equation reduces to the well known Lyapunov equation considered also in [1,2].

Substituting $\mathbf{F} = \mathbf{A}^T$ for the simplicity of notation and assuming that the matrix $\mathbf{F}$ is defined by the following decomposition

$$\mathbf{F} = \mathbf{K} - \mathbf{L} - \mathbf{U}, \qquad (2)$$

where $\mathbf{K}$, $\mathbf{L}$ and $\mathbf{U}$ are nonsingular diagonal, strictly lower triangular and strictly upper triangular parts of $\mathbf{A}$ respectively; Eq.(1) can be rewritten as follows

$$\mathbf{K}\mathbf{X} = \mathbf{L}\mathbf{X} + \mathbf{U}\mathbf{X} - \mathbf{X}\mathbf{A} + \mathbf{X}\mathbf{S}\mathbf{X} - \mathbf{Q} \qquad (3)$$

or equivalently

$$\mathbf{X} = \mathbf{K}^{-\mathbf{1}}\{\mathbf{L}\mathbf{X} + \mathbf{U}\mathbf{X} - \mathbf{X}\mathbf{A} + \mathbf{X}\mathbf{S}\mathbf{X} - \mathbf{Q}\}. \qquad (4)$$

We assume that the iterative scheme has the following form

$$\mathbf{X}^{(t)} = \mathbf{K}^{-\mathbf{1}}\{\mathbf{L}\mathbf{X}^{(t)} + \mathbf{U}\mathbf{X}^{(t-1)} - \mathbf{X}^{(t-1,t)}\mathbf{A} + \mathbf{X}^{(t-1,t)}\mathbf{S}\mathbf{X}^{(t-1,t)} - \mathbf{Q}\} \quad \text{for } t = 1, 2, \ldots, \qquad (5)$$

where the terms $\mathbf{X}^{(t-1,t)}\mathbf{A}$ and $\mathbf{X}^{(t-1,t)}\mathbf{S}\mathbf{X}^{(t-1,t)}$ mean that for computing the products $\mathbf{X}\mathbf{A}$ and $\mathbf{X}\mathbf{S}\mathbf{X}$, the entries of $\mathbf{X}^{(t-1)}$ and $\mathbf{X}^{(t)}$ are used in the current iteration t.

For the acceleration of convergence in the above iterative scheme, the overrelaxation procedure can be used as follows

$$\begin{aligned}\mathbf{X}^{(t)} &= \omega\mathbf{K^{-1}}\{\mathbf{L}\mathbf{X}^{(t)} + \mathbf{U}\mathbf{X}^{(t-1)} - \mathbf{X}^{(t-1,t)}\mathbf{A} + \mathbf{X}^{(t-1,t)}\mathbf{S}\mathbf{X}^{(t-1,t)} - \mathbf{Q}\} \\ &- (\omega-1)\mathbf{X}^{(t-1)} \quad \text{for } t = 1,2,\ldots \end{aligned} \tag{6}$$

or written equivalently

$$\begin{aligned}\mathbf{X}^{(t)} &= [\mathbf{I} - \omega\mathbf{K^{-1}}\mathbf{L}]^{\mathbf{-1}}\{[(1-\omega)\mathbf{I} + \omega\mathbf{K^{-1}}\mathbf{U}]\mathbf{X}^{(t-1)} \\ &+ \omega\mathbf{K}^{-1}[-\mathbf{X}^{(t-1,t)}\mathbf{A} + \mathbf{X}^{(t-1,t)}\mathbf{S}\mathbf{X}^{(t-1,t)} - \mathbf{Q}]\} \text{ for } t = 1,2,\ldots. \end{aligned} \tag{7}$$

Since the exact solution $\mathbf{X}$ satisfies the above equation, then with the error solution matrix $\mathbf{E}^{(t)} = \mathbf{X} - \mathbf{X}^{(t)}$, we have

$$\begin{aligned}\mathbf{E}^{(t)} &= [\mathbf{I} - \omega\mathbf{K^{-1}}\mathbf{L}]^{\mathbf{-1}}\{[(1-\omega)\mathbf{I} + \omega\mathbf{K^{-1}}\mathbf{U}]\mathbf{E}^{(t-1)} \\ &+ \omega\mathbf{K}^{-1}[-\mathbf{E}^{(t-1,t)}\mathbf{A} + \mathbf{E}^{(t-1,t)}\mathbf{S}\mathbf{E}^{(t-1,t)} - \mathbf{Q}]\} \text{ for } t = 1,2,\ldots. \end{aligned} \tag{8}$$

Similarly as in the case of Sylvester equation solutions analyzed in [1,2], the error solution matrix $\mathbf{E}^{(t)}$ can not be expressed explicitly in dependence on $\mathbf{E}^{(0)}$ however, we can assume that there exists an implict iteration matrix $\mathcal{T}_{impl}$ which form can not be expessed explicitly but we are able to compute its spectral radius according to the following equation derived from (6)

$$\begin{aligned}&\omega\mathbf{K^{-1}}\{\mathbf{L}\mathbf{Y}^{(t)} + \mathbf{U}\mathbf{Y}^{(t-1)} - \mathbf{Y}^{(t-1,t)}\mathbf{A} + \mathbf{Y}^{(t-1,t)}\mathbf{S}\mathbf{Y}^{(t-1,t)}\} \\ &\qquad - (\omega-1)\mathbf{Y}^{(t-1)} = \Lambda\mathbf{Y}^{(t)} \quad \text{for } t = 1,2,\ldots, \end{aligned} \tag{9}$$

where Λ is an eigenvalue of the implicit iteration matrix $\mathcal{T}_{impl}$ and the matrix $\mathbf{Y}$ plays a role of an "eigenvector". When Λ is a real eigenvalue, the above equation represents the algorithm of the power method providing us the spectral radius $\varrho(\mathcal{T}_{impl}) = |\Lambda|$, where the value of $\varrho(\mathcal{T}_{impl})$ is frequently minimized for $0 < \omega < 1$ and method is not sensitive to the choice of the accurate value of ω_{best}.

As is observed in numerical experiments presented in the paper, the proposed method, based on a simple algorithm, provides very acurate solutions with a low computational work because the arithmetical effort in one iteration is roughly equivalent to that required for computing the residual matrix

$$\mathbf{R}^{(t)} = \mathbf{F}(\equiv \mathbf{A}^T)\mathbf{X}^{(t)} + \mathbf{X}^{(t)}\mathbf{A} - \mathbf{X}^{(t)}\mathbf{G}\mathbf{X}^{(t)} + \mathbf{Q}. \tag{10}$$

Keywords

Algebraic Riccati equation, SOR-LIKE iterative method, Spectral radius, Implicit iteration matrix.

References:

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